FROBENIUS EXTENSIONS AND WEAK HOPF ALGEBRAS

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Abstract. We study a symmetric Markov extension of k-algebras $N \hookrightarrow M$, a certain kind of Frobenius extension with conditional expectation that is tracial on the centralizer and dual bases with a separability property. We place a depth two condition on this extension, which is essentially the requirement that the Jones tower $N \hookrightarrow M \hookrightarrow M_1 \hookrightarrow M_2$ can be obtained by taking relative tensor products with centralizers $A = C_{M_1}(N)$ and $B = C_{M_2}(M)$. Under this condition, we prove that $N \hookrightarrow M$ is the invariant subalgebra pair of a weak Hopf algebra action by A, i.e., that $N = M^A$. The endomorphism algebra $M_1 = \mathrm{End}_N M$ is shown to be isomorphic to the smash product algebra $M\#A$. We also extend results of Szymański [26], Vainerman and the second author [18], and the authors [11].

1. Introduction and Preliminaries

In its most general setting, the Jones tower is the iteration of the endomorphism ring construction over any non-commutative ring extension $S \hookrightarrow R_0$, which results in a tower of rings over R_0 [8]. The first step is to form $R_0 \hookrightarrow R_1 := \mathrm{End}^r_S R_0$ via the left regular representation. The process may then be repeated to obtain $R_1 \hookrightarrow R_2 := \mathrm{End}_{R_0} R_1$. For a finite index subfactor [7] or a Markov extension [10] $N \subseteq M = M_0$ the algebras in the Jones tower have their usual form $M_n = M_{n-1} e_n M_{n-1}$ for $n = 1, 2, 3, \dots$ where e_n are the Jones idempotents. Up to Morita equivalence of rings, the Jones tower over a Markov extension has periodicity two.

Now *weak* Hopf algebras generalize Hopf algebras and are Hopf-like objects with self-dual axioms, introduced by Böhm and Szlachányi in [3] and in [2] with Nill. It is well understood now that Hopf algebras and weak Hopf algebras arise as non-commutative symmetries of Jones towers of certain finite index inclusions of topological algebras over the complex numbers. For a finite index von Neumann subfactor $N \subseteq M$ it was shown by Szymański [26] that the depth 2 condition for the associated tower of centralizers $C_M(N) \subseteq C_{M_1}(N) \subseteq C_{M_2}(M) \subseteq \cdots$ is equivalent to $A := C_{M_1}(N)$ having a natural structure of a Hopf C^*-algebra, if $C_M(N) = \mathbb{C}1$. In the general case where $C_M(N) \supseteq \mathbb{C}1$, it was shown by Vainerman and the second author [18] that the depth two condition is equivalent to A being a *weak* Hopf C^*-algebra. In both cases, A acts on M in such a way that $N = M^A$ and $M_1 \cong M\#A$; moreover, $B := C_{M_2}(M)$ is naturally identified with the (weak) Hopf C^*-algebra

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dual to A. An outline of the proof of an analogous result for depth 2 inclusions of unital C^*-algebras was given very recently by Szlachányi [25].

In [19] it was shown that a finite index and finite depth II_1 subfactor is embeddable in a weak Hopf C^*-algebra smash product inclusion; whence it is canonically determined via a Galois-type correspondence by some weak Hopf C^*-algebra and its coideal $*$-subalgebra. In this respect, weak Hopf C^*-algebras play the same role as Ocneanu's paragroups [21].

In [11] hypotheses of depth 2 are placed on a Markov extension $N \subseteq M$ of algebras over a field k with trivial centralizer $C_M(N) = k1$ such that $A = C_{M_1}(N)$ can be given a semisimple Hopf algebra structure via the Szymański pairing [26]. Moreover, A acts on M in such a way that the Jones tower above M is isomorphic to a duality-for-actions tower obtained from the smash product of M and A and the standard left action of A^* on A:

$$
\begin{array}{ccccccc}
N & \hookrightarrow & M & \hookrightarrow & M_1 & \hookrightarrow & M_2 \\
\| & & \| & & \downarrow \cong & & \downarrow \cong \\
M^A & \hookrightarrow & M & \hookrightarrow & M\#A & \hookrightarrow & M\#A\#A^*.
\end{array}
\tag{1}
$$

We can continue iteration in the isomorphic copy of the Jones tower by alternately acting by A and its dual A^*. Indeed, it is a well-known theorem in algebra and operator algebras that the algebra $M\#A\#A^*$ above is isomorphic to the endomorphism algebra $\mathrm{End}(M\#A)_M$ (cf. [1] for Hopf algebras and [16] for weak Hopf algebras).

In this paper, we extend (1) to a Markov extension $N \hookrightarrow M$ which satisfies less restrictive conditions than trivial centralizer and free extension M_1/M as in [11]. We assume conditions slightly stronger than $U := C_M(N)$ being a separable algebra on which the Markov trace T is non-degenerate. For the depth 2 conditions, we assume that the canonical conditional expectations E_M and E_{M_1} have dual bases in A and its dual centralizer $B = C_{M_2}(M)$, respectively. In exchange we obtain a canonical structure of a semisimple and cosemisimple weak Hopf algebra on A with the dual B. Furthermore, the smash products above no longer have k-vector space structure given by $M\#A = M \otimes_k A$ and $M\#A\#A^* \cong M_1 \otimes_k B$, but by $M\#A = M \otimes_U A$ and $M\#A\#A^* \cong M_1 \otimes_V B$, where $V = C_{M_1}(M)$.

This paper is organized as follows.

In this section we move on to cover preliminaries essential to this paper — weak Hopf algebras and their actions, Markov extensions, the Basic Construction Theorem, and conditions of symmetry and weak irreducibility on Markov extensions that will be needed in the later sections.

In Section 2 we place depth 2 conditions on the Jones tower over a symmetric and weakly irreducible Markov extension, and develop a series of propositions and lemmas on depth 2 properties on the centralizers $U \subseteq A \subseteq C = C_{M_2}(N)$ and $V \subseteq B \subseteq C$, in both cases, C being the basic construction for Markov extensions of the same index as M/N.

In Sections 3 and 4 we show that A is a weak Hopf algebra with the action on M outlined above. First, in Section 3 we place a coalgebra structure on B by defining a non-degenerate pairing with A; the antipode $S : B \to B$ comes from a symmetry in the definition of the pairing. The rest of this section is devoted to proving that this structure on B satisfies the axioms of a weak Hopf algebra. It follows that A is the dual weak Hopf algebra of B. Second, in Section 4 an action of B on

M_1 is introduced, and two equivalent expressions for this action are given. Then we establish a left action of A on M with the outcome as in (1): the two vertical isomorphisms following from Theorems (4.6) and (4.3) together with Propositions (4.1) and (4.5), which establish the actions of A and its dual.

We note here that the main results in [11, Sections 1-6] are recovered in this paper if U is trivial. Furthermore, the results of this paper may be viewed as an answer to the question implicit in [2, last line, p. 387]. In an appendix, we extend to Markov extensions the Pimsner-Popa formula for the Jones idempotent generating the basic construction of composites in a Jones tower, and also give a special example of a depth 2 algebra extension.

Weak Hopf algebras. Throughout this paper we work over an arbitrary field k and use the Sweedler notation for a comultiplication on a coalgebra H, writing $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$.

The following definition of a weak Hopf algebra and related notions were introduced in an equivalent form by Böhm and Szlachányi in [3] (see also [2]). We refer the reader to the recent survey [20] for an introduction to the theory of weak Hopf algebras and its applications.

Definition 1.1 ([3], [2])**.** A *weak Hopf algebra* is a k-vector space H that has structures of an algebra $(H, m, 1)$ and a coalgebra (H, Δ, ε) such that the following axioms hold:

1. Δ is a (not necessarily unit-preserving) algebra homomorphism:

$$\Delta(hg) = \Delta(h)\Delta(g). \tag{2}$$

2. The unit and counit satisfy the identities:

$$\varepsilon(hgf) = \varepsilon(hg_{(1)})\varepsilon(g_{(2)}f) = \varepsilon(hg_{(2)})\varepsilon(g_{(1)}f), \tag{3}$$

$$(\Delta \otimes \mathrm{id})\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{4}$$

3. There exists a linear map $S : H \to H$, called an *antipode*, satisfying the following axioms:

$$m(\mathrm{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \tag{5}$$

$$m(S \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)), \tag{6}$$

$$S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h), \tag{7}$$

for all $h, g, f \in H$.

Axioms (3) and (4) are analogous to the bialgebra axioms specifying ε as an algebra homomorphism and Δ a unit-preserving map, and Axioms (5) and (6) generalize the properties of the antipode with respect to the counit. In addition, it may be shown that given (2) - (6), Axiom (7) is equivalent to S being both an algebra and coalgebra anti-homomorphism.

A *morphism* of weak Hopf algebras is a map between them which is both an algebra and a coalgebra morphism commuting with the antipode.

Below we summarize the basic properties of weak Hopf algebras, see [2], [20] for the proofs.

The antipode S of a weak Hopf algebra H is unique; if H is finite-dimensional then it is bijective.

The right-hand sides of the formulas (5) and (6) are called the *target* and *source counital maps* and denoted ε_t, ε_s respectively:

$$\varepsilon_t(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \tag{8}$$

$$\varepsilon_s(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)). \tag{9}$$

The counital maps ε_t and ε_s are idempotents in $\mathrm{End}_k(H)$, and satisfy relations $S \circ \varepsilon_t = \varepsilon_s \circ S$ and $S \circ \varepsilon_s = \varepsilon_t \circ S$.

The main difference between weak and usual Hopf algebras is that the images of the counital maps are not necessarily equal to $k1_H$. They turn out to be subalgebras of H called *target* and *source counital subalgebras* or *bases* as they generalize the notion of a base of a groupoid (cf. examples below):

$$H_t = \{h \in H \mid \varepsilon_t(h) = h\} = \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in H^*\}, \tag{10}$$

$$H_s = \{h \in H \mid \varepsilon_s(h) = h\} = \{(\mathrm{id} \otimes \phi)\Delta(1) \mid \phi \in H^*\}. \tag{11}$$

The counital subalgebras commute with each other and the restriction of the antipode gives an algebra anti-isomorphism between H_t and H_s.

The algebra H_t (resp. H_s) is separable (and, therefore, semisimple) with the separability idempotent $e_t = (S \otimes \mathrm{id})\Delta(1)$ (resp. $e_s = (\mathrm{id} \otimes S)\Delta(1)$).

Note that H is an ordinary Hopf algebra if and only if $\Delta(1) = 1 \otimes 1$, iff ε is a homomorphism, and iff $H_t = H_s = k1_H$.

When $\dim_k H < \infty$, the dual vector space $H^* = \mathrm{Hom}_k(H, k)$ has a natural structure of a weak Hopf algebra with the structure operations dual to those of H:

$$\langle \phi\psi, h \rangle = \langle \phi \otimes \psi, \Delta(h) \rangle, \tag{12}$$

$$\langle \Delta(\phi), h \otimes g \rangle = \langle \phi, hg \rangle, \tag{13}$$

$$\langle S(\phi), h \rangle = \langle \phi, S(h) \rangle, \tag{14}$$

for all $\phi, \psi \in H^*$, $h, g \in H$. The unit of H^* is ε and the counit is $\phi \mapsto \langle \phi, 1 \rangle$.

Example 1.2. Let G be a *groupoid* over a finite base (i.e., a category with finitely many objects, such that each morphism is invertible), then the groupoid algebra kG is generated by morphisms $g \in G$ with the unit $1 = \sum_X \mathrm{id}_X$, where the sum is taken over all objects X of G, and the product of two morphisms is equal to their composition if the latter is defined and 0 otherwise. It becomes a weak Hopf algebra via:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G. \tag{15}$$

The counital maps are given by $\varepsilon_t(g) = gg^{-1} = \mathrm{id}_{\mathrm{target}(g)}$ and $\varepsilon_s(g) = g^{-1}g = \mathrm{id}_{\mathrm{source}(g)}$.

If G is finite then the dual weak Hopf algebra $(kG)^*$ is generated by idempotents p_g, $g \in G$ such that $p_g p_h = \delta_{g,h} p_g$ and

$$\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,gg^{-1}} = \delta_{g,g^{-1}g}, \quad S(p_g) = p_{g^{-1}}. \tag{16}$$

It is known that any group action on a set gives rise to a finite groupoid. Similarly, in the non-commutative situation, one can associate a weak Hopf algebra with every action of a usual Hopf algebra on a separable algebra, see [17] for details. More interesting examples of weak Hopf algebras arise from dynamical twisting of Hopf algebras [4], closely related to the quantum dynamical Yang-Baxter equation, and from the applications to the subfactor theory ([18], [19]).

Definition 1.3 ([2], 3.1)**.** A left (right) *integral* in H is an element $l \in H$ $(r \in H)$ such that

$$hl = \varepsilon_t(h)l, \qquad (rh = r\varepsilon_s(h)) \qquad \text{for all } h \in H. \tag{17}$$

These notions clearly generalize the corresponding notions for Hopf algebras ([15], 2.1.1). We denote $\int_H^l$ (respectively, $\int_H^r$) the space of left (right) integrals in H and by $\int_H = \int_H^l \cap \int_H^r$ the space of two-sided integrals.

An integral in H (left or right) is called *non-degenerate* if it defines a non-degenerate functional on H^*. A left integral l is called *normalized* if $\varepsilon_t(l) = 1$. Similarly, $r \in \int_H^r$ is normalized if $\varepsilon_s(r) = 1$. The Maschke theorem for weak Hopf algebras [2] states that a weak Hopf algebra H is semisimple if and only if it is separable, and if and only if it has a normalized integral. In particular, every semisimple weak Hopf algebra is finite dimensional.

Example 1.4. (i) Let G^0 be the set of units of a finite groupoid G, then the elements $l_e = \sum_{gg^{-1}=e} g\,(e \in G^0)$ span $\int_{kG}^l$ and elements $r_e = \sum_{g^{-1}g=e} g\,(e \in G^0)$ span $\int_{kG}^r$.
(ii) If $H = (kG)^*$ then $\int_H^l = \int_H^r = \mathrm{span}\{p_e |\, e \in G^0\}$.

Definition 1.5. An algebra A is a (left) *H-module algebra* if A is a left H-module via $h \otimes a \to h \cdot a$ and

1) $h \cdot ab = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,
2) $h \cdot 1 = \varepsilon_t(h) \cdot 1$.

If A is an H-module algebra we will also say that H acts on A. The invariants $A^H = \{a \in A |\, h \cdot a = \varepsilon_t(h) \cdot a, \forall\, h \in H\}$ form a subalgebra by 2) above and a calculation involving [2, (2.8a),(2.7a)].

Definition 1.6. An algebra A is a (right) *H-comodule algebra* if A is a right H-comodule via $\rho : a \mapsto a^{(0)} \otimes a^{(1)}$ and

1) $\rho(ab) = a^{(0)}b^{(0)} \otimes a^{(1)}b^{(1)}$,
2) $\rho(1) = (\mathrm{id} \otimes \varepsilon_t)\rho(1)$.

The coinvariants $A^{\mathrm{co}H} = \{a \in A |\, \rho(a) = a^{(0)} \otimes \varepsilon_t(a^{(1)})\}$ form a subalgebra of A.

It follows immediately that A is a left H-module algebra if and only if A is a right H^*-comodule algebra.

Example 1.7.

(i) The target counital subalgebra H_t is a trivial H-module algebra via $h \cdot z = \varepsilon_t(hz)$, $h \in H$, $z \in H_t$.
(ii) H is an H^*-module algebra via the dual, or standard, action $\phi \rightharpoonup h = h_{(1)}\langle \phi, h_{(2)} \rangle$, $\phi \in H^*$, $h \in H$.
(iii) Let $A = C_H(H_s) = \{a \in H \mid ay = ya \quad \forall y \in H_s\}$ be the centralizer of H_s in H, then A is an H-module algebra via the adjoint action $h \cdot a = h_{(1)}aS(h_{(2)})$.

Let A be an H-module algebra, then a *smash product* algebra $A\#H$ is defined on a k-vector space $A \otimes_{H_t} H$, where H is a left H_t-module via multiplication and A is a right H_t-module via

$$a \cdot z = S^{-1}(z) \cdot a = a(z \cdot 1), \qquad a \in A, z \in H_t,$$

as follows. Let $a\#h$ be the class of $a \otimes h$ in $A \otimes_{H_t} H$, then the multiplication in $A\#H$ is given by the familiar formula

$$(a\#h)(b\#g) = a(h_{(1)} \cdot b)\#h_{(2)}g, \qquad a, b, \in A,\ h, g \in H,$$

and the unit of $A\#H$ is $1\#1$.

The smash product $A\#H$ is a left H^*-module algebra via

$$\phi \cdot (a\#h) = a\#(\phi \rightharpoonup h), \quad \phi \in H^*,\ h \in H,\ a \in A.$$

It was shown in [16] that there is a canonical isomorphism of algebras $(A\#H)\#H^* \cong \mathrm{End}(A\#H)_A$, which extends the well-known duality theorem for usual Hopf algebras [1].

Symmetric Markov extensions. Again let k be a ground field. Recall that an algebra extension M/N is *Frobenius* if there is an N-bimodule homomorphism $E : M \to N$ and elements $\{x_i\}$, $\{y_i\}$ in M such that for all $m \in M$,

$$E(mx_i)y_i = m = x_iE(y_im), \tag{18}$$

where a summation over repeated indices is understood throughout the paper. We refer to E, $\{x_i\}$, $\{y_i\}$ as Frobenius coordinates, E being called a *Frobenius homomorphism*, and the elements $\{x_i\}$, $\{y_i\}$ are called *dual bases*. Another set of Frobenius coordinates $F : M \to N$, $\{r_j\}$, $\{\ell_j\}$ is related to the first by $F = Ed$ and *dual bases tensor* by $e = r_j \otimes \ell_j = x_i \otimes d^{-1}y_i$ where $d = F(x_i)y_i$ is in the centralizer $C_M(N)$ [13, 22, 27]. Note that e is a Casimir element, i.e., satisfies $me = em$ for all $m \in M$ by a computation as in Lemma (1.8) below. A Frobenius homomorphism E is left *non-degenerate* (or faithful) in the sense that $E(xM) = 0$ implies $x = 0$; similarly, E is right non-degenerate. Being Frobenius is a transitive property of extensions with respect to the composition of Frobenius homomorphisms.

An algebra extension M'/N' is said to be *split* if N' is isomorphic to an N'-bimodule direct summand in M'. For example, a Frobenius extension M/N is split if there is $d \in C_M(N)$ such that $E(d) = 1$ in the notation above, since Ed is then a bimodule projection $M \to N$.

A Frobenius extension M/N is *symmetric* if there is a Frobenius homomorphism E such that $Eu = uE$ for each $u \in C_M(N)$; i.e., $E(ux) = E(xu)$ for all $x \in M, u \in C_M(N)$ [14]. Let $U = C_M(N)$ for the rest of this section. For example, the symmetry condition is satisfied by a symmetric algebra A/k [29]. As an application of the symmetry condition, we have:

Lemma 1.8. *For all $u \in U$,*

$$x_iu \otimes y_i = x_i \otimes uy_i \tag{19}$$

in $M \otimes_N M$.

Proof. We compute using Eqs. (18):

$$x_iu \otimes y_i = x_jE(y_jx_iu) \otimes y_i = x_j \otimes E(uy_jx_i)y_i = x_j \otimes uy_j. \quad \square$$

Recall that a Frobenius extension M/N is *strongly separable* if $E(1) = 1$ and $x_iy_i = \lambda^{-1}1 \in k1$ for some $\lambda \in k^\circ$ whose reciprocal is called the *index*, denoted by $\lambda^{-1} = [M : N]_E$ [9, 10]. (In the terminology of [27], E is a conditional expectation with quasi-basis x_i, y_i such that the index of E is nonzero in $k1$.) We say that a strongly separable extension is a *Markov extension* if there is a (Markov) trace $T : N \to k$ such that $T(1) = 1_k$ and $T_0 := T \circ E$ is a trace on M [9, 10]. A Frobenius homomorphism E that is a trace-preserving bimodule projection is referred to as a *conditional expectation*.

Definition 1.9. We refer to an extension of algebras M/N as a *symmetric Markov extension* if it is a Markov extension with coordinates E, $\{x_i\}$, $\{y_i\}$ and Markov trace T such that for each $u \in U$, $E(ux) = E(xu)$ for every $x \in M$.

For example, the symmetric Frobenius condition is satisfied by the irreducible Markov extensions in [11], since U is trivial for these. As another example, the symmetric Frobenius condition is satisfied when T is non-degenerate on N, e.g., for a II_1 subfactor $N \subseteq M$ of finite index [7]:

Proposition 1.10. *If the Markov trace T is non-degenerate on N, then $uE = Eu$ for every $u \in U$.*

Proof. We note that: for all $n \in N, m \in M$

$$T(nE(um)) = T_0(num) = T_0(unm) = T_0(nmu) = T(nE(mu)),$$

which implies that $E(um) = E(mu)$ for all $m \in M$. □

Let $M_1 = M \otimes_N M \cong \mathrm{End}(M_N)$ denote the basic construction of M/N: i.e., $M_1 = Me_1M$ where $e_1 = 1 \otimes 1$ is the first Jones idempotent with conditional expectation $E_M : M_1 \to M$ given by $E_M(me_1m') = \lambda mm'$, dual bases $\{\lambda^{-1}x_ie_1\}$, $\{e_1y_i\}$, and index-reciprocal λ. Recall that $M_1 \cong \mathrm{End}(M_N)$ is given by $me_1m' \mapsto \ell_m E \ell_{m'}$ where ℓ_m is left multiplication by $m \in M$. The E-multiplication induced by composition on $\mathrm{End}(M_N)$ is given by

$$e_1me_1 = e_1E(m) = E(m)e_1$$

for all $m \in M$.

Theorem 1.11 ("Basic Construction")**.** *Suppose $N \hookrightarrow M$ is a symmetric Markov extension. Then M_1 is a symmetric Markov extension of M with Markov trace $T_0 = T \circ E$ and is characterized by having idempotent e_1 and conditional expectation $E_M : M_1 \to M$ such that*

1. *$M_1 = Me_1M$;*
2. *$E_M(e_1) = \lambda 1$;*
3. *for each $x \in M$: $e_1xe_1 = e_1E(x) = E(x)e_1$.*

Proof. Most of the proof is found in [9] or [10]: we need only establish the symmetric Frobenius condition as well as the characterization above.

Let $V = C_{M_1}(M)$. Note that U is anti-isomorphic to V as algebras, via the map

$$\phi : U \to V, \quad \phi(u) = x_iue_1y_i, \tag{20}$$

which has inverse given by $v \mapsto \lambda^{-1}E_M(ve_1)$. Clearly then $V \cong U^{\mathrm{op}}$. Note also that

$$E_M(ve_1) = E_M(e_1v)$$

as a consequence of Lemma (1.8).

We compute that $E_Mv = vE_M$ for all $\phi(u) \in V$ and all $a, b \in M$:

$$E_M(\phi(u)ae_1b) = E_M(x_iue_1y_iae_1b) = E_M(x_iE(y_ia)ue_1b) = \lambda aub,$$

while

$$E_M(ae_1b\phi(u)) = E_M(ae_1bx_iue_1y_i) = E_M(ae_1E(bx_iu)y_i) = \lambda aub.$$

Suppose $\tilde{M}$ is an algebra with idempotent f and conditional expectation $\tilde{E} : \tilde{M} \to M$ satisfying the conditions above. Since $\tilde{M} = MfM$ and $nf = fn$ for each $n \in N$, there is a surjective mapping of $M_1 \to \tilde{M}$. If $xf = fx$ for some

$x \in M$, then $fxf = fE(x) = fx$; applying $\tilde{E}$ and the Condition (2), we see that $x = E(x) \in N$. It follows that the mapping $M_1 \to \tilde{M}$ is an algebra isomorphism forming a commutative triangle with $\tilde{E}$ and E_M. □

Let us recall that a k-algebra A is *Kanzaki separable* (also called strongly separable algebra in the literature) if it has a symmetric separability element, or equivalently, if the trace of the left regular representation of A on itself has dual bases $\{x_i\}$ and $\{y_i\}$ such that $x_i y_i = 1$ [12]. Yet another equivalent condition: A is k-separable with invertible Hattori-Stalling rank as a finitely generated projective module over its center [24]. For example, the full p-by-p matrix algebra over a characteristic p field F is separable but not Kanzaki separable. Over a non-perfect field F, a separable F-algebra is in turn finite dimensional semisimple, but not necessarily the converse. In characteristic zero, all three notions coincide.

For the rest of this paper, we will make the two assumptions below on a symmetric Markov extension M/N.

1. (Symmetric product assumption.) $x_i y_i = y_i x_i = \lambda^{-1} 1 \in k1$.
2. (Weak irreducible assumption.) U is a Kanzaki separable k-algebra with non-degenerate trace $T_0|_U$.

Under these assumptions, it follows from the proof of the basic construction theorem that V is also Kanzaki separable. The next proposition shows that $T_1 := TEE_M$ is a non-degenerate trace on V.

Lemma 1.12. *We have the identity $T_1 \circ \phi = T_0$ on U.*

Proof. Let $u \in U$. We compute using the symmetric product assumption:

$$T_1\phi(u) = T_1(x_i u e_1 y_i) = \lambda T_0(x_i u y_i) = \lambda T_0(y_i x_i u) = T_0(u). \quad \square$$

Remark 1.13. For the purposes of this paper, the symmetric product assumption may be replaced by the identity in the statement of Lemma (1.12). This last condition holds trivially for an irreducible Markov extension as in [11].

Since M_1/M is also a symmetric Markov extension with index λ^{-1}, we now iterate the basic construction to form $M_2 = M_1 e_2 M_1$ with conditional expectation $E_{M_1}(xe_2y) = \lambda xy$ for each $x, y \in M_1$ and *second Jones idempotent* e_2. We recall the braid-like relations,

$$e_1 e_2 e_1 = \lambda e_1$$

and

$$e_2 e_1 e_2 = \lambda e_2$$

established in [10], and the Pimsner-Popa relations,

$$xe_1 = \lambda^{-1} E_M(xe_1) e_1 \quad \forall\, x \in M_1, \tag{21}$$

and three more similar equations for $e_1 x$, $e_2 y$ and $y e_2$ where $y \in M_2$ [11].

2. Properties of depth 2 extensions

Let M/N be a symmetric Markov extension satisfying the weak irreducible condition and the symmetric product condition in Section 1. Recall that this entails three conditions on a Markov extension $(E : M \to N, x_i, y_i, \lambda, T : N \to k)$:

1. $E : M \to N$ is symmetric: $Eu = uE$ for each $u \in U = C_M(N)$.
2. U is Kanzaki separable and $T_0|_U$ is a non-degenerate trace.

$$\begin{array}{ccccccccccccc}
 & & & & & & C & & & & & & \\
 & & & & & / & & \backslash & & & & & \\
 & & & & A & & & & B & & & & \\
 & & & / & & \backslash & & / & & \backslash & & & \\
 & & U & & & & V & & & & W & & \\
 & / & & \backslash & & / & & \backslash & & / & & \backslash & \\
N & & \hookrightarrow & & M & & \hookrightarrow & & M_1 & & \hookrightarrow & & M_2
\end{array}$$

FIGURE 1. Hasse Diagram for Centralizers.

3. $y_i x_i = \lambda^{-1} = x_i y_i$; alternatively, $T_0|_U = T_1 \circ \phi$ where $\phi : U \to V$ is the anti-isomorphism defined in Eq. (20).

In this section, we work with the Jones tower above M/N:

$$N \overset{\xleftarrow{E}}{\hookrightarrow} M \overset{\xleftarrow{E_M}}{\hookrightarrow} M_1 \overset{\xleftarrow{E_{M_1}}}{\hookrightarrow} M_2. \tag{22}$$

We denote the "second" centralizers by $A = C_{M_1}(N)$, $B = C_{M_2}(M)$, and the "big" centralizer by $C = C_{M_2}(N)$, which contains A, B. Note that U and V are contained in A; V and $W = C_{M_2}(M_1)$ are contained in B; and $V = A \cap B$. See Figure 1.

Definition 2.1. We say that M/N has a (weak) *depth* 2 property if the following conditions are satisfied by its Jones tower:

1. E_M has dual bases $\{z_j\}$, $\{w_j\}$ in A.
2. E_{M_1} has dual bases $\{u_i\}$, $\{v_i\}$ in B.

We note that the depth 2 conditions in [11] are a special case of these. However, the weak depth 2 conditions may depend on the choice of conditional expectation $E : M \to N$.

Remark 2.2. If M/N is a subfactor of a finite index von Neumann factor (i.e., $[M : N] < \infty$) then the above notion of depth 2 coincides with the usual one.

Note that the definition of depth 2 makes sense for a Frobenius extension M/N, since for these we retain an endomorphism ring theorem stating that Frobenius coordinates E, x_i, y_i for M/N lead to coordinates $E_M(me_1m') = mm'$ $(m, m' \in M)$ with dual bases $\{x_i e_1\}$, $\{e_1 y_i\}$ for $M_1 = M \otimes_N M \cong \mathrm{End}(M_N)$ as a Frobenius extension over M [22]. (However, we no longer necessarily have $E(1) = 1$ and $e_1^2 = e_1$.)

We will denote by T the restriction of the normalized trace $T_2 = T_1 E_{M_1}$ of M_2 on C.

Lemma 2.3. *A, B are separable algebras with $T|_A, T|_B$ as non-degenerate traces.*

Proof. From the first of the depth 2 conditions, we see that $E_M(az_j)w_j = a = z_j E_M(w_j a)$ for all $a \in A \subset M_1$. Since $z_j w_j = \lambda^{-1}1$ and $E_M(A) = U$, we readily see that A is a strongly separable extension of U with Markov trace of index λ^{-1}.

Similarly, B/V is a strongly separable extension with $E_{M_1} : B \to V$ as conditional expectation, dual bases $\{u_i\}$, $\{v_i\}$ and index λ^{-1}. In particular, A is a separable extension of the separable algebra U, and is itself a separable algebra [6]. Similarly, B is k-separable. $T|_A$ is a non-degenerate trace on A since it is a Frobenius homomorphism by transitivity: $T|_A = T|_U \circ E_M|_A$ by the Markov property. Similarly, $T|_B$ is a non-degenerate trace. □

Lemma 2.4. *As vector spaces, $M_2 \cong M_1 \otimes_V B$ via the mapping $m_1 \otimes b \mapsto m_1 b$. Similarly, $M_1 \cong M \otimes_U A$.*

Proof. The inverse mapping is given by $x \mapsto E_{M_1}(xu_j) \otimes v_j$. We note that

$$E_{M_1}(ybu_j) \otimes v_j = y \otimes E(bu_j)v_j = y \otimes b$$

for $y \in M_1, b \in B$, since $E_{M_1}(B) = V$. The second statement can be proven similarly. □

We develop the following depth 2 properties for the algebra extension M/N above in a series of propositions. We let $E_A = E_{M_1}|_C$.

Proposition 2.5 (Existence of E_B)**.** *There exists a B-bimodule map $E_B : C \to B$ such that $E_B|_B = id_B$, E_B is a conditional expectation such that $E_B(e_1) = \lambda 1$ and $T(E_B(c)b) = T(bc)$ for all $b \in B$ and $c \in C$.*

Proof. Let $\{a_i\}$, $\{b_i\}$ denote dual bases in U for T restricted thereon. It follows from Lemma (1.12) that the elements $\{c_i := \phi(a_i)\}$, $\{d_i := \phi(b_i)\}$ are dual bases for the trace T restricted to V. Define E_B by

$$E_B(c) = T(cu_jc_i)d_iv_j. \tag{23}$$

Since $\{u_jc_i\}$, $\{d_iv_j\}$ are dual bases for $T = TE_{M_1} : B \to k$ by transitivity, it follows that $E_B(b) = b$ and $E_B(cb) = E_B(c)b$ for every $b \in B$. The left B-module property of E_B follows from: for all $b \in B, c \in C$,

$$E_B(bc) = T(bcu_jc_i)d_iv_j = T(cu_jc_ib)d_iv_j = T(cu_jc_i)bd_iv_j,$$

since $u_jc_ib \otimes d_iv_j = u_jc_i \otimes bd_iv_j$ by Lemma (1.8).

Next,

$$T(E_B(c)) = T(cu_jc_i)T(d_iv_j) = T(c)$$

since $u_jc_iT(d_iv_j) = 1$.

Finally, let $F = E_ME_A$ and use the Pimsner-Popa relations as well as the expression for ϕ^{-1} below Eq. (20) to compute:

$$\begin{aligned} E_B(e_1) = T(e_1u_jc_i)d_iv_j &= T(e_1E_A(u_j)c_i)d_iv_j \\ &= \lambda^{-1}T(e_1F(e_1u_j)c_i)d_iv_j \\ &= T(\lambda^{-1}E_M(e_1c_i)F(e_1u_j))d_iv_j \\ &= x_ke_1T(E_M(e_1(E_A(u_j))a_i)b_iy_kv_j \\ &= \lambda x_ke_1E_A(u_j)v_jy_k = \lambda 1_{M_1}. \quad \square \end{aligned}$$

Proposition 2.6 ("Commuting square condition")**.** *We have $E_A \circ E_B = E_B \circ E_A$.*

Proof. We compute: for each $c \in C$,

$$E_A E_B(c) = T(cu_j c_i) d_i E_A(v_j) = T(cu_j E_A(v_j) c_i) d_i = T(cc_i) d_i,$$

while

$$E_B E_A(c) = T(E_A(c) E_A(u_j) c_i) d_i v_j = T(E_A(c) c_i) d_i E_A(u_j) v_j = T(cc_i) d_i$$

by the Markov property $T E_A = T$. □

Proposition 2.7 ("Symmetric square condition")**.** *We have $AB = BA = C$. More precisely, $A \otimes_V B \cong B \otimes_V A \cong C$ as vector spaces.*

Proof. We note that $E_A(C) = A$ and $V = A \cap B$. The proposition follows easily from the dual bases equations and the depth 2 assumption:

$$E_A(cu_j) v_j = c = u_j E_A(v_j c),$$

for all $c \in C$. □

Proposition 2.8 (Pimsner-Popa identities)**.** *We have*

$$\begin{array}{rcll} \lambda^{-1} e_2 E_A(e_2 c) & = & e_2 c, & \lambda^{-1} E_A(c e_2) e_2 = c e_2, \\ \lambda^{-1} e_1 E_B(e_1 c) & = & e_1 c, & \lambda^{-1} E_B(c e_1) e_1 = c e_1. \end{array}$$

As a consequence we have

$$\begin{array}{rcll} Ce_2 & = & Ae_2, & e_2 C = e_2 A, \\ Ce_1 & = & Be_1, & e_1 C = e_1 B. \end{array}$$

Proof. Now $e_2 C = e_2 A$ and $Ce_2 = Ae_2$ follow from the usual Pimsner-Popa Eqs. (21) for $E_{M_1}|_C = E_A$. At a point below in this proof, we will need to know that

$$C = A e_2 A, \tag{24}$$

which follows from

$$c = E_A(cu_j) v_j = \lambda^{-1} E_A(c z_i e_2) e_2 w_i,$$

for by the basic construction theorem $u_j \otimes v_j = \lambda^{-1} z_i e_2 \otimes e_2 w_i$ in $M_2 \otimes_{M_1} M_2$.

Note that $F(C) = U$. We compute: for each $c \in C$,

$$\begin{array}{rcl} e_1 c = e_1 E_A(cu_j) v_j & = & \lambda^{-1} e_1 E_M(e_1 E_A(cu_j)) v_j \\ & = & \lambda^{-1} e_1 T(F(e_1 c u_j) a_i) b_i v_j \\ & = & \lambda^{-3} T(c u_j E_M(c_i e_1) e_1) e_1 E_M(e_1 d_i) v_j \\ & = & \lambda^{-1} T(c u_j c_i e_1) e_1 d_i v_j = \lambda^{-1} e_1 E_B(e_1 c). \end{array}$$

Thus, $e_1 C = e_1 B$.

The computation $ce_1 = \lambda^{-1} E'_B(ce_1) e_1$ proceeds similarly, where

$$E'_B(c) = u_j c_i T(d_i v_j c), \tag{25}$$

clearly defines a bimodule projection of C onto B (cf. Proposition (2.5)). As a result, we have $Ce_1 = Be_1$.

We will show that $E_B = E'_B$ by showing that $C = Be_1 B$ and noting that $E'_B(e_1) = \lambda 1$ by a computation very similar to that for $E_B(e_1) = \lambda 1$ above. Using the braid-like relations and Eq. (24), we compute:

$$C = Ae_2 A = Ae_2 e_1 e_2 A = Ce_1 C = Be_1 B. \quad \square$$

It is not hard to show that $E_B : C \to B$ is isomorphic to the basic construction of the strongly separable extension B/V, where $C = Be_1B$. Similarly, $E_A : C \to A$ is isomorphic to the basic construction of the strongly separable extension A/U, where $C = Ae_2A$.

Remark 2.9. Irreducible separable Markov extensions considered in [11] trivially satisfy the weak irreducibility assumption as well as the conclusion of Lemma (1.12). It follows that all the results of the next sections apply to these.

3. Weak Hopf algebra structures on centralizers

Let $f = f^{(1)} \otimes f^{(2)}$ be the unique symmetric separability element [24] of $V = C_{M_1}(M)$, and let $w = [f^{(1)}T(f^{(2)})]^{-1} \in Z(V)$ be the invertible element satisfying $f^{(1)}T(vwf^{(2)}) = v$ for all $v \in V$. In other words, $f^{(1)} \otimes wf^{(2)}$ is the dual bases tensor for $T : V \to k$.

Proposition 3.1. *The following bilinear form,*

$$\langle\, a,\, b \,\rangle = \lambda^{-2} T(ae_2e_1wb), \qquad a \in A,\, b \in B,$$

is non-degenerate on $A \otimes B$.

Proof. If $\langle\, a,\, B \,\rangle = 0$ for some $a \in A$, then for all $x \in C$ we have $T(ae_2e_1x) = 0$, since $e_1B = e_1C$ (depth 2 property). Taking $x = e_2a'\,(a' \in A)$ and using the braid-like relation between Jones idempotents, and Markov property of T we have

$$T(aa') = \lambda^{-1} T(ae_2e_1(e_2a')) = 0 \qquad \text{for all } a' \in A,$$

therefore $a = 0$. Similarly, one proves that $\langle\, A,\, b \,\rangle = 0$ implies $b = 0$. □

The above duality form allows us to introduce a comultiplication $b \mapsto b_{(1)} \otimes b_{(2)}$ on B as follows:

$$\langle\, a_1,\, b_{(1)} \,\rangle \langle\, a_2,\, b_{(2)} \,\rangle = \langle\, a_1a_2,\, b \,\rangle, \tag{26}$$

for all $a_1, a_2 \in A$, $b \in B$, and counit $\varepsilon : B \to k$ by $(\forall b \in B)$

$$\varepsilon(b) = \langle\, 1,\, b \,\rangle. \tag{27}$$

A proof similar to that of Proposition (3.1) shows that $\langle\, a, b \,\rangle' = \lambda^{-2}T(be_1e_2wa)$ is another non-degenerate pairing of A and B. We then introduce a linear automorphism $S : B \to B$ by the relation $\langle\, a, b \,\rangle = \langle\, a, S(b) \,\rangle'$, i.e.,

$$\langle\, a, b \,\rangle = \lambda^{-2} T(S(b)e_1e_2wa) \tag{28}$$

for all $a \in A$, $b \in B$, or, equivalently,

$$E_A(e_2e_1wb) = E_A(S(b)e_1e_2)w. \tag{29}$$

Note that we automatically have

$$E_{M_1}(e_2xwb) = E_{M_1}(S(b)xe_2)w, \qquad \text{for all } x \in M_1. \tag{30}$$

Proposition 3.2. *We note that: (for all $b, c \in B$)*

$$\varepsilon(b) = \lambda^{-1} T(e_2wb), \tag{31}$$

$$\varepsilon(S(b)) = \varepsilon(b), \tag{32}$$

$$\Delta(1) = S^{-1}(f^{(1)}) \otimes f^{(2)}. \tag{33}$$

Proof. The formula for ε follows from the identity $E_B(e_1) = \lambda 1$ and $T \circ E_B = T$:

$$\varepsilon(b) = \lambda^{-2} T(e_2 e_1 w b) = \lambda^{-1} T(e_2 w b).$$

Then the second equation follows from the computation :

$$\varepsilon(b) = \lambda^{-1} T(e_2 w b) = \lambda^{-2} T(b E_B(e_1) e_2 w) = \lambda^{-2} T(e_2 e_1 w S^{-1}(b)) = \varepsilon(S^{-1}(b)).$$

To establish the third formula, we use the Markov property and commuting square condition to compute: for all $a, a' \in A$,

$$\begin{aligned} \langle a, S^{-1}(f^{(1)}) \rangle \langle a', f^{(2)} \rangle &= \lambda^{-3} T(a e_2 e_1 w S^{-1}(f^{(1)})) T(E_A \circ E_B(a' e_1 w) f^{(2)}) \\ &= \lambda^{-3} T(f^{(1)} e_1 e_2 w a) T(E_B(a' e_1) w f^{(2)}) \\ &= \lambda^{-2} T(E_B(a' e_1) e_1 e_2 w a) \\ &= \lambda^{-2} T(a a' e_1 e_2 w) = \langle a a', 1 \rangle. \quad \square \end{aligned}$$

The following lemma gives a useful explicit formula for S^{-1}.

Lemma 3.3. *For all $b \in B$ we have $S^{-1}(b) = \lambda^{-3} w^{-1} E_B(e_1 e_2 E_A(b e_1 e_2)) w$.*

Proof. We obtain this formula by multiplying both sides of Eq. (29) by $e_1 e_2$ on the left and taking E_B from both sides. $\square$

Corollary 3.4. *We have $S(V) = W$, where $W = C_{M_2}(M_1)$.*

Proof. Let us take $y \in W$, then using Lemma (3.3), the commuting square condition, and the Markov property we have

$$\begin{aligned} S^{-1}(y) &= \lambda^{-3} w^{-1} E_B(e_1 e_2 e_1 E_A(y e_2)) w \\ &= \lambda^{-2} w^{-1} E_B(e_1 E_A(y e_2)) w \in V. \end{aligned}$$

Therefore, $S^{-1}(W) \subseteq V$ and since $W \cong V$ as vector spaces, we have $S(V) = W$. $\square$

Lemma 3.5. *For all $b \in B$ we have $b = w S^{-1}(w S^{-1}(b) w^{-1}) w^{-1}$.*

Proof. Using non-degeneracy of the duality form and definition of S we compute for all $a \in A$:

$$\begin{aligned} T(a e_2 e_1 b) &= \lambda^{-1} T(E_A(b a e_2) e_2 e_1) \\ &= \lambda^{-1} T(E_A(e_2 a w S^{-1}(b)) w^{-1} e_2 e_1) \\ &= T(e_2 a w S^{-1}(b) w^{-1} e_1) \\ &= T(E_A(w S^{-1}(b) w^{-1} e_1 e_2) w w^{-1} a) \\ &= T(a E_A(e_2 e_1 w S^{-1}(w S^{-1}(b) w^{-1})) w^{-1}), \end{aligned}$$

whence the formula follows. $\square$

Proposition 3.6. *S is an algebra anti-homomorphism, i.e.,*

$$S(b b') = S(b') S(b) \qquad \text{for all } b, b' \in B.$$

Proof. We use the non-degeneracy of the duality form:

$$\begin{aligned} T(a e_2 e_1 w S^{-1}(b') w^{-1} S^{-1}(b)) &= \lambda^{-1} T(w^{-1} E_A(S^{-1}(b) a e_2) e_2 e_1 w S^{-1}(b')) \\ &= \lambda^{-1} T(E_A(w^{-1} e_2 a w S^{-2}(b)) w^{-1} e_2 e_1 w S^{-1}(b')) \\ &= \lambda^{-1} T(b' e_1 e_2 E_A(e_2 a w S^{-2}(b)) w^{-1}) \\ &= T(w S^{-2}(b) w^{-1} b' e_1 e_2 a w) \\ &= T(a e_2 e_1 w S^{-1}(w S^{-2}(b) w^{-1} b') w^{-1}), \end{aligned}$$

therefore, we have $S^{-1}(b')w^{-1}S^{-1}(b)w = S^{-1}(wS^{-2}(b)w^{-1}b')$. Using Lemma (3.5) we conclude that

$$S^{-1}(b')S^{-1}(wS^{-2}(b)w^{-1}) = S^{-1}(b')w^{-1}S^{-1}(b)w = S^{-1}(wS^{-2}(b)w^{-1}b').$$

We replace $wS^{-2}(b)w^{-1}$ by b to obtain the result. □

Corollary 3.7. *For all $b \in B$ we have $S^2(b) = gbg^{-1}$ where*

$$g := S(w^{-1})w. \tag{34}$$

In particular, $S^2|_V = \mathrm{id}_V$ from (3.4), so S maps V to W and vice versa, as well as $S^2|_W = \mathrm{id}_W$. For example, we obtain $\Delta(1) = S(f^{(1)}) \otimes f^{(2)}$ from this and (3.2).

Lemma 3.8. *For all $b \in B$ and $v \in V$ we have*

$$\Delta(bv) = \Delta(b)(v \otimes 1). \tag{35}$$

Proof. Let $a, a' \in A$ then

$$\begin{aligned} \langle\, a \otimes a',\, \Delta(bv)\,\rangle &= \langle\, aa',\, bv\,\rangle = \langle\, vaa',\, b\,\rangle \\ &= \langle\, a,\, b_{(1)}v\,\rangle\langle\, a',\, b_{(2)}\,\rangle. \quad \square \end{aligned}$$

Now we are in the position to establish the unit and counit axioms for B.

Proposition 3.9. *We have*

$$(id \otimes \Delta)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{36}$$

Proof. We have seen that $\Delta(1) \in W \otimes V$, therefore $(1 \otimes \Delta(1))$ and $(\Delta(1) \otimes 1)$ commute. By Lemma (3.8),

$$\begin{aligned} (1 \otimes \Delta(1))(\Delta(1) \otimes 1) &= S^{-1}(f^{(1)}) \otimes 1_{(1)}f^{(2)} \otimes 1_{(2)} \\ &= S^{-1}(f^{(1)}) \otimes \Delta(f^{(2)}) = (\mathrm{id} \otimes \Delta)\Delta(1). \quad \square \end{aligned}$$

Proposition 3.10. *For all $b, c, d \in B$ we have*

$$\varepsilon(bcd) = \varepsilon(bc_{(1)})\varepsilon(c_{(2)}d) = \varepsilon(bc_{(2)})\varepsilon(c_{(1)}d).$$

Proof. First, one can define a coalgebra structure on A using the duality form from Proposition (3.1) and show that $\Delta_A(1) \in A \otimes C_M(N)$ in a way similar to how it was shown above for the comultiplication Δ of B that $\Delta(1) \in W \otimes V$. Then we compute:

$$\begin{aligned} \varepsilon(bcd) &= \lambda^{-1}T(e_2wbcd) \\ &= \lambda^{-3}T(E_A(de_2)e_2e_1wbc) \\ &= \langle\, 1_{(1)},\, b\,\rangle\langle\, \lambda^{-1}E_A(de_2)1_{(2)},\, c\,\rangle \\ &= \langle\, 1_{(1)},\, b\,\rangle\langle\, 1_{(2)},\, c_{(2)}\,\rangle\langle\, \lambda^{-1}E_A(de_2),\, c_{(1)}\,\rangle \\ &= \varepsilon(bc_{(2)})\varepsilon(c_{(1)}d). \end{aligned}$$

Note that in the third line $E_A(de_2)$ commutes with each of the elements in $\{1_{(2)}\} \subset U$, so that $\varepsilon(bcd)$ is also equal to $\varepsilon(bc_{(1)})\varepsilon(c_{(2)}d)$. □

The next step is to prove that Δ is a homomorphism. To achieve this we first need to establish a certain commutation relation (see Proposition (3.13) below) that corresponds to the two different ways of representing $C = AB = BA$.

We will need several preliminary results.

Lemma 3.11. *The following identities hold for all $b \in B$ and $v \in V$:*

(a) $S^{-1}(e_2) = w^{-1}e_2 w$,
(b) $ve_2 = S(v)e_2$,
(c) $\lambda^{-1}E_A(e_2wb)w^{-1} = \varepsilon(b1_{(1)})1_{(2)}$,
(d) $\Delta(b)(1 \otimes v) = \Delta(b)(S(v) \otimes 1)$,
(e) $\Delta(b)\Delta(1) = \Delta(b)$.

Proof. (a) We have $T(ae_2e_1wS^{-1}(e_2)) = T(e_2e_1e_2wa) = T(ae_2e_1e_2w)$, whence the result follows by non-degeneracy of the bilinear pairing $a \otimes b \mapsto T(ae_2e_1b)$.

(b) We compute, using part (a) and the anti-multiplicativity of S:

$$\begin{aligned}\lambda^2\langle\, a,\, S^{-1}(ve_2)\,\rangle &= T(ve_2e_1e_2wa)\\ &= T(ae_2e_1wS^{-1}(ve_2))\\ &= \lambda T(ae_2wS^{-1}(v))\\ &= T(S^{-1}(v)e_2e_1e_2wa) = \lambda^2\langle\, a,\, S^{-1}(S^{-1}(v)e_2)\,\rangle.\end{aligned}$$

(c) Since both sides of the given equation belong to V, it suffices to evaluate them against $T(\cdot\, v)$ for all $v \in V$:

$$\begin{aligned}T(\lambda^{-1}E_A(e_2wb)v) &= \lambda^{-1}T(e_2wbv) = \lambda^{-1}T(ve_2wb)\\ T(\varepsilon(b1_{(1)})1_{(2)}wv) &= \varepsilon(bS(f^{(1)}))T(vwf^{(2)})\\ &= \varepsilon(bS(v)) = \lambda^{-1}T(e_2wbS(v))\\ &= \lambda^{-1}T(ve_2wb),\end{aligned}$$

where we used part (b).

(d) We evaluate both sides against elements $a \otimes a' \in A \otimes A$ (note that $S(v)$ commutes with A):

$$\begin{aligned}\langle\, a \otimes a',\, b_{(1)}S(v) \otimes b_{(2)}\,\rangle &= \lambda^{-2}T(S(v)ae_2e_1wb_{(1)})\langle\, a',\, b_{(2)}\,\rangle\\ &= \lambda^{-2}T(ave_2e_1wb_{(1)})\langle\, a',\, b_{(2)}\,\rangle\\ &= \langle\, av,\, b_{(1)}\,\rangle\langle\, a',\, b_{(2)}\,\rangle = \langle\, ava',\, b\,\rangle\\ &= \langle\, a \otimes a',\, b_{(1)} \otimes b_{(2)}v\,\rangle.\end{aligned}$$

(e) From part (d), properties of S, and properties of the separability element f we have

$$\begin{aligned}\Delta(b)\Delta(1) &= b_{(1)}1_{(1)} \otimes b_{(2)}1_{(2)} = b_{(1)}S(1_{(2)})1_{(1)} \otimes b_{(2)}\\ &= b_{(1)}S(f^{(1)}f^{(2)}) \otimes b_{(2)} = b_{(1)} \otimes b_{(2)}. \quad \square\end{aligned}$$

Applying S to part (a) above, we obtain from part (b):

$$S(e_2) = w^{-1}e_2w. \tag{37}$$

Proposition 3.12. *For all $a \in A$ and $b \in B$ we have*

(i) $\lambda^{-1}E_B(e_1wba) = \langle\, a,\, b_{(1)}\,\rangle wb_{(2)}$,
(ii) $\lambda^{-1}b_{(2)}E_A(e_2wb_{(1)})w^{-1} = b$.

Proof. (i) Let $a' \in A$ then

$$\begin{aligned}\langle\, a',\, \lambda^{-1}w^{-1}E_B(e_1wba)\,\rangle &= \lambda^{-3}T(a'e_2e_1E_B(e_1wba'))\\ &= \lambda^{-2}T(a'e_2e_1wba') = \langle\, aa',\, b\,\rangle\\ &= \langle\, a',\, \langle\, a,\, b_{(1)}\,\rangle b_{(2)}\,\rangle.\end{aligned}$$

(ii) From Lemma (3.11c) and (e) we have

$$\lambda^{-1}b_{(2)}E_A(e_2wb_{(1)})w^{-1} = \varepsilon(b_{(1)}1_{(1)})b_{(2)}1_{(2)} = b. \quad \square$$

The next Proposition (cf. [11], 4.6) is the key ingredient in proving that B is a weak Hopf algebra acting on M_1.

Proposition 3.13. *For all $b \in B$ we have*

$$w^{-1}e_1wb = \lambda^{-1}b_{(2)}w^{-1}E_A(e_2e_1wb_{(1)}). \tag{38}$$

Proof. First, let us note that for all $c_1, c_2 \in C$ we have $c_1 = c_2$ if and only if $E_B(c_1a) = E_B(c_2a)$ for all $a \in A$. Indeed, if $c \in C$ and $E_B(ca) = 0$ for all $a \in A$ then $T(abc) = T(bE_B(ca)) = 0$ for all $b \in B$. But since $AB = C$ by Proposition (2.7) and T is non-degenerate, we conclude that $c = 0$.

Let $c_1 = w^{-1}e_1wb$ and $c_2 = \lambda^{-1}b_{(2)}w^{-1}E_A(e_2e_1wb_{(1)})$. We compute, using Propositions (3.12) and (2.6) :

$$\begin{aligned}
E_B(c_1a) &= w^{-1}E_B(e_1wba) = w^{-1}\langle\, a,\, b_{(1)}\,\rangle wb_{(2)} \\
&= \langle\, a,\, b_{(1)}\,\rangle b_{(2)}, \\
E_B(c_2a) &= \lambda^{-1}b_{(2)}w^{-1}E_B\circ E_A(e_2e_1wb_{(1)}a) \\
&= \lambda^{-1}b_{(2)}w^{-1}E_A(e_2E_B(e_1wb_{(1)}a)) \\
&= \lambda^{-1}\langle\, a,\, b_{(1)}\,\rangle b_{(3)}w^{-1}E_A(e_2wb_{(2)}) \\
&= \langle\, a,\, b_{(1)}\,\rangle b_{(2)},
\end{aligned}$$

whence the result follows. $\square$

Corollary 3.14. *For all $b \in B$ and $x \in M_1$ we have*

$$w^{-1}xb = \lambda^{-1}b_{(2)}w^{-1}E_{M_1}(e_2xb_{(1)}). \tag{39}$$

Proof. This follows from the fact that every $x \in M_1$ can be written as $x = \sum x_ie_1y_i$, where $x_i, y_i \in M$ commute with B. $\square$

Corollary 3.15. *For all $x, y \in M_1$ and $b \in B$, we have*

$$E_{M_1}(e_2wyxb) = \lambda^{-1}E_{M_1}(e_2wyb_{(2)})w^{-1}E_{M_1}(e_2wxb_{(1)}). \tag{40}$$

Proof. This is obtained from Corollary (3.14) by replacing x with wx, multiplying both sides by e_2wy on the left, and taking E_A from both sides. $\square$

In order to prove the multiplicativity of Δ we first need to establish anti-comultiplicativity of S.

Proposition 3.16. *S is anti-comultiplicative, i.e.,*

$$\Delta S(b) = S(b_{(2)}) \otimes S(b_{(1)}) \qquad \textit{for all } b \in B. \tag{41}$$

Proof. Let $a, a' \in A$ then using Corollary (3.15) and Lemma (3.11d) we compute:

$$\begin{aligned}
\langle\, aa',\, S^{-1}(b)\,\rangle &= \lambda^{-3}T(e_1e_2E_A(e_2waa'b))\\
&= \lambda^{-4}T(e_1e_2E_A(e_2wab_{(2)})w^{-1}E_A(e_2wa'b_{(1)}))\\
&= \lambda^{-2}\langle\, w^{-1}E_A(e_2wab_{(2)})w^{-1}E_A(e_2wa'b_{(1)}),\, 1\,\rangle\\
&= \lambda^{-2}\langle\, w^{-1}E_A(e_2wab_{(2)}),\, 1_{(1)}\,\rangle\langle\, w^{-1}E_A(e_2wa'b_{(1)}),\, 1_{(2)}\,\rangle\\
&= \lambda^{-6}T(S(1_{(1)})e_1e_2E_A(e_2wab_{(2)}))T(S(1_{(2)})e_1e_2E_A(e_2wa'b_{(1)}))\\
&= \lambda^{-4}T(b_{(2)}S(1_{(1)})e_1e_2wa)T(b_{(1)}S(1_{(2)})e_1e_2wa')\\
&= \langle\, a,\, S^{-1}(b_{(2)}S(1_{(1)}))\,\rangle\langle\, a',\, S^{-1}(b_{(1)}S(1_{(2)}))\,\rangle\\
&= \langle\, a,\, S^{-1}(b_{(2)})\,\rangle\langle\, a',\, S^{-1}(b_{(1)}1_{(1)}S(1_{(2)}))\,\rangle\\
&= \langle\, a,\, S^{-1}(b_{(2)})\,\rangle\langle\, a',\, S^{-1}(b_{(1)})\,\rangle,
\end{aligned}$$

since $f^{(2)}f^{(1)} = 1$, whence the proposition follows from non-degeneracy of $\langle\ ,\ \rangle$ and bijectivity of S. ☐

Proposition 3.17. Δ *is a homomorphism of algebras:*

$$\Delta(bb') = \Delta(b)\Delta(b') \qquad \text{for all } b, b' \in B. \tag{42}$$

Proof. Using the definition and properties of S and Corollary (3.15) for all $x, y \in M_1$ we have:

$$\begin{aligned}
E_{M_1}(S(b)xw^{-1}ye_2)w &= E_{M_1}(e_2xw^{-1}ywb)\\
&= \lambda^{-1}E_{M_1}(e_2xb_{(2)})w^{-1}E_{M_1}(e_2ywb_{(1)})\\
&= \lambda^{-1}E_{M_1}(S(b_{(2)})xw^{-1}e_2)E_{M_1}(S(b_{(1)})ye_2)w,
\end{aligned}$$

and using Corollary (3.16) and bijectivity of S we obtain:

$$E_{M_1}(bxye_2) = \lambda^{-1}E_{M_1}(b_{(1)}xe_2)E_{M_1}(b_{(2)}ye_2) \qquad \text{for all } x, y \in M_1, b \in B. \tag{43}$$

Next, using the duality form we have: for $a, a' \in A$,

$$\begin{aligned}
\langle\, a\otimes a',\, \Delta(bb')\,\rangle &= \langle\, aa',\, bb'\,\rangle\\
&= \lambda^{-1}\langle\, E_A(b'aa'e_2), b\,\rangle\\
&= \lambda^{-2}\langle\, E_A(b'_{(1)}ae_2), b_{(1)}\,\rangle\langle\, E_A(b'_{(2)}ae_2), b_{(2)}\,\rangle\\
&= \langle\, a,\, b_{(1)}b'_{(1)}\,\rangle\langle\, a',\, b_{(2)}b'_{(2)}\,\rangle,
\end{aligned}$$

as required. ☐

Next we establish properties of the antipode with respect to the counital maps.

Proposition 3.18. *For all $b \in B$ we have the following identities:*

$$S(b_{(1)})b_{(2)} = 1_{(1)}\varepsilon(b1_{(2)}), \tag{44}$$

$$b_{(1)}S(b_{(2)}) = \varepsilon(1_{(1)}b)1_{(2)}. \tag{45}$$

Proof. To establish the first relation we compute, using Eq. (43), for all $a \in A$:

$$\begin{aligned}
\langle\, a,\, S^{-1}(b_{(1)})w^{-1}b_{(2)}\,\rangle &= \lambda^{-1}\langle\, E_A(w^{-1}b_{(2)}ae_2),\, S^{-1}(b_{(1)})\,\rangle\\
&= \lambda^{-4}T(E_A(w^{-1}b_{(2)}ae_2)e_2E_A(e_2e_1wS^{-1}(b_{(1)})))\\
&= \lambda^{-3}T(E_A(b_{(2)}ae_2)E_A(b_{(1)}e_1e_2))\\
&= \lambda^{-2}T(be_1ae_2).
\end{aligned}$$

Next we recall the formula for $\Delta(1)$ from Proposition (3.2), formula for S^2 from Corollary (3.7), Lemma (3.11d), and that $\Delta(w) = \Delta(1)(w \otimes 1) = (w \otimes 1)\Delta(1)$:

$$\begin{aligned}
\langle\, a,\, 1_{(1)}\varepsilon(b1_{(2)})\,\rangle &= \lambda^{-1}\langle\, a,\, 1_{(1)}\,\rangle T(e_2wb1_{(2)}) \\
&= \lambda^{-1}\langle\, a,\, S^{-1}(f^{(1)})\,\rangle T(e_2wbf^{(2)}) \\
&= \lambda^{-1}\langle\, a,\, S^{-1}(E_A(e_2wbw^{-1}))\,\rangle \\
&= \lambda^{-3}T(E_A(e_2wbw^{-1})e_1e_2wa) \\
&= \lambda^{-2}T(e_2wbw^{-1}e_1wa) \\
&= \langle\, wa,\, S^{-1}((wbw^{-1})_{(1)})w^{-1}(wbw^{-1})_{(2)}\,\rangle \\
&= \langle\, a,\, S^{-1}(wb_{(1)}w^{-1})w^{-1}b_{(2)}w\,\rangle \\
&= \langle\, a,\, S^{-1}(wS(w^{-1})b_{(1)}S(w)w^{-1})b_{(2)}\,\rangle \\
&= \langle\, a,\, S(b_{(1)})b_{(2)}\,\rangle.
\end{aligned}$$

The second identity follows from the first by (3.2), since the symmetry of f and the anti-(co)multiplicative properties of the antipode imply :

$$\begin{aligned}
b_{(1)}S(b_{(2)}) &= S(S(S^{-1}(b)_{(1)})S^{-1}(b)_{(2)}) = S(1_{(1)})\varepsilon(S^{-1}(b)1_{(2)}) \\
&= \varepsilon(S(1_{(2)})b)S(1_{(1)}) = \varepsilon(1_{(1)}b)1_{(2)}. \quad \square
\end{aligned}$$

Let us consider two mappings $\varepsilon_t : B \to V$ and $\varepsilon_s : B \to W$ given by $\varepsilon_t(b) = \varepsilon(1_{(1)}b)1_{(2)}$, and $\varepsilon_s(b) = 1_{(1)}\varepsilon(b1_{(2)})$, corresponding to the right-hand side of the equations in Proposition (3.18). They are called the *target and source counital maps*, respectively (cf. Section 1). By a computation quite similar to that in Lemma (3.11c), we may check that:

$$\varepsilon_t(b) = \lambda^{-1}E_A(be_2). \tag{46}$$

Indeed, we have for each $v \in V$,

$$T(\varepsilon(1_{(1)}b)1_{(2)}v) = \varepsilon(S(vw^{-1})b) = \lambda^{-1}T(e_2wS(w^{-1})S(v)b) = \lambda^{-1}T(e_2vb)$$

while also $T(\lambda^{-1}E_A(be_2)v) = \lambda^{-1}T(e_2vb)$.

Theorem 3.19. *$(B, \Delta, \varepsilon, S)$ is a semisimple weak Hopf algebra.*

Proof. Semisimplicity follows from Lemma (2.3). We have established all the axioms of a weak Hopf algebra except Axiom (7), which we show next. At a point below, we let $b' = S(b)$, at another $b'' = wb'$, and use Eq. (39) as well as Lemma (3.8). Let $g = S(w^{-1})w$ be the element from Corollary (3.7) implementing the inner automorphism S^2, then for all $b \in B$,

$$\begin{aligned}
S(b_{(1)})b_{(2)}S(b_{(3)}) &= \lambda^{-1}S(b_{(1)})E_A(b_{(2)}e_2) \\
&= \lambda^{-1}b'_{(2)}E_A(S^{-1}(b'_{(1)})e_2) \\
&= \lambda^{-1}b'_{(2)}E_A(e_2wg^{-1}b'_{(1)}g)w^{-1} \\
&= \lambda^{-1}b'_{(2)}E_A(e_2wb'_{(1)}S(w^{-1})) \\
&= \lambda^{-1}b''_{(2)}w^{-1}E_A(e_2b''_{(1)}) = w^{-1}b'' = S(b). \quad \square
\end{aligned}$$

Remark 3.20. (i) $V = \varepsilon_t(B)$ is the target counital subalgebra of B and $W = C_{M_2}(M_1) = S(V)$ is the source counital subalgebra (recall that the antipode maps one counital subalgebra to another).

(ii) From Eq. (46) we see that e_2 is a normalized left integral in B:

$$be_2 = \lambda^{-1} E_A(be_2)e_2 = \varepsilon_t(b)e_2.$$

Furthermore, $l = e_2 S^{-1}(e_2) = e_2 w^{-1} e_2 w = e_2 w$ is a two-sided integral in B, due to Lemma (3.11a) and the fact that the space of left (respectively, right) integrals in a weak Hopf algebra is a left (respectively, right) ideal. Next, $S(l) = w^{-1} S(w) e_2 w = e_2 w = l$, since $\varepsilon_t|_W = S|_W$. Finally l is normalized, since

$$\varepsilon_t(l) = \lambda^{-1} E_A(E_M(w)e_1) = 1 \quad \text{and} \quad \varepsilon_s(l) = S \circ \varepsilon_t(l) = 1.$$

Clearly, l is the unique element with these properties (cf. [18, 5.7]). Such a two-sided normalized integral is called a *Haar integral* in [2].

Defining a comultiplication and counit of A similarly to Eqs. (26) and (27), as the dual of the multiplication and unit of B, and an antipode S_A on A by $\langle S_A(a), b \rangle = \langle a, S(b) \rangle$, the corollary below follows from the self-duality of the axioms of a weak Hopf algebra and Lemma (2.3).

Corollary 3.21. *A is a semisimple weak Hopf algebra isomorphic to the dual of B.*

4. Action and smash product

In this section we define an action of B on M_1 suggested by the measuring in Eq. (43), and show that it comes from the standard left action of a weak Hopf algebra on its dual. We then show that M is the subalgebra of invariants of this action, and that M_2 is isomorphic to the smash product of M_1 with B.

Proposition 4.1. *The mapping $\triangleright : B \otimes M_1 \to M_1$ given by*

$$b \triangleright x = \lambda^{-1} E_{M_1}(bxe_2) \tag{47}$$

defines a left action of a weak Hopf algebra on M_1, characterized by

$$b \triangleright ma = m \langle a_{(2)}, b \rangle a_{(1)} \tag{48}$$

for all $m \in M, a \in A, b \in B$. In particular, M is the subalgebra of invariants for this action.

Proof. From Eq. (43) it follows that $\triangleright$ satisfies the measuring axiom. From Eq. (46) it follows that $b \triangleright 1 = \varepsilon_t(b)$. The action of B on M_1 is a left module action of an algebra by the Pimsner-Popa relations and $E_{M_1}(xe_2) = \lambda x$ for $x \in M_1$.

Recall that $M_1 = MA$. Since $B = C_{M_2}(M)$, it is clear that $b \triangleright ma = m(b \triangleright a)$ for every $m \in M$. We compute for every $a \in A, b, b' \in B$:

$$\langle a_{(1)}, b' \rangle \langle a_{(2)}, b \rangle = \langle a, b'b \rangle = \langle \lambda^{-1} E_A(bae_2), b' \rangle = \langle b \triangleright a, b' \rangle,$$

whence Eq. (48) follows. Thus the action of B on A coincides with the standard left action of a weak Hopf algebra B on its dual $B^* \cong A$ as in Example (1.7(ii)). Since the invariant subalgebra A^B is $k1$, it follows that $M_1^B = M$. □

The next proposition provides a simplifying formula for this action. We will need the equation

$$b_{(1)} S(b_{(2)}) b_{(3)} = b \tag{49}$$

for each $b \in B$, which follows from Eq. (45).

Proposition 4.2. *For every $b \in B, x \in M_1$, we have*

$$b \triangleright x = b_{(1)} x S(b_{(2)}).$$

Proof. We subsequently use Eq. (39), Lemma (3.11d) and its opposite (obtained by applying $S \otimes S$), Corollary (3.7), and Eq. (49) in the next computation: for every $b \in B, x \in M_1$,

$$\begin{aligned} b_{(1)} x S(b_{(2)}) &= \lambda^{-1} b_{(1)} w S(b_{(2)}) w^{-1} E_{M_1}(e_2 x S(b_{(3)})) \\ &= \lambda^{-1} b_{(1)} S(b_{(2)}) E_{M_1}(e_2 x S(w^{-1} b_{(3)} w)) \\ &= \lambda^{-1} \varepsilon_t(b_{(1)}) E_{M_1}(S(w^{-1}) b_{(2)} x e_2) w \\ &= \lambda^{-1} E_{M_1}(S(w^{-1}) b x e_2) w. \end{aligned}$$

Next note that $\Delta(v') = (1 \otimes v')\Delta(1)$ for all $v' \in W$, which follows from an application of S to Lemma (3.8). Then let $b' = S(w^{-1})b$ and compute:

$$b' \triangleright x = (S(w)b')_{(1)} x S((S(w)b')_{(2)}) w^{-1} = b'_{(1)} x S(S(w) b'_{(2)}) w^{-1} = b'_{(1)} x S(b'_{(2)}). \quad \square$$

Theorem 4.3. *The mapping $\psi : x\#b \mapsto xb \in M_2$ defines an isomorphism between the algebra M_2 and the smash product algebra $M_1\#B$.*

Proof. That ψ is a linear isomorphism follows from Lemma (2.4).

That ψ is a homomorphism follows almost directly from Eq. (49) and the conjugation formula in Proposition (4.2):

$$bx = b_{(1)} x \varepsilon_s(b_{(2)}) = (b_{(1)} \triangleright x) b_{(2)},$$

since for all $b' \in B$: $\varepsilon_s(b') = S(b'_{(1)}) b'_2 \in W = C_{M_2}(M_1)$. $\square$

Action of A on M. In this subsection, we define a left action of A on M by a formula similar to that for $\triangleright$ of B in Proposition (4.2). Denote the antipode of A by S below. Let ε_s and ε_t again denote the source and target counital maps on A.

Lemma 4.4. *The map ε_t is a non-unital module homomorphism ${}_AA \to {}_{\mathrm{ad}}A$ with respect to the left regular and adjoint actions of A on itself: for all $a, a' \in A$,*

$$a_{(1)} \varepsilon_t(a') S(a_{(2)}) = \varepsilon_t(aa').$$

The proof of this and a similar fact for $\varepsilon_s : A_A \to A_{\mathrm{ad}}$ is easy and omitted.

Proposition 4.5. *The mapping $\triangleright : A \otimes M \to M$ given by*

$$a \triangleright m = a_{(1)} m S(a_{(2)}) \tag{50}$$

is a weak Hopf algebra action of A on M.

Proof. First we check that $a \triangleright m \in M$ given $m \in M, a \in A$. Let $\rho : M_1 \to M_1 \otimes A$, $\rho(x) = x^{(0)} \otimes x^{(1)}$, denote the coaction dual to the action $B \otimes M_1 \to M_1$ above. Then $b \triangleright x = x^{(0)}\langle x^{(1)}, b\rangle$. It follows from Eq. (48) that ρ restricted to A is the comultiplication:

$$a^{(0)} \otimes a^{(1)} = a_{(1)} \otimes a_{(2)}.$$

Since M is shown above to be the invariant subalgebra of this action of B on M_1, it is also precisely the coinvariant subalgebra of ρ. We then compute using Lemma (4.4):

$$\begin{aligned} \rho(a \triangleright m) &= a_{(1)} m^{(0)} S(a_{(4)}) \otimes a_{(2)} \varepsilon_t(m^{(1)}) S(a_{(3)}) \\ &= a_{(1)} m^{(0)} S(a_{(3)}) \otimes \varepsilon_t(a_{(2)} m^{(1)}) \\ &= (a \triangleright m)^{(0)} \otimes \varepsilon_t((a \triangleright m)^{(1)}), \end{aligned}$$

whence $a \triangleright m \in M$.

Since $\varepsilon_s(A) = V = C_{M_1}(M)$, we compute that $\triangleright$ measures M:

$$\begin{aligned}(a_{(1)} \triangleright m)(a_{(2)} \triangleright m') &= a_{(1)} m S(a_{(2)}) a_{(3)} m' S(a_{(4)}) \\ &= a_{(1)} \varepsilon_s(a_{(2)}) m m' S(a_{(3)}) \\ &= a \triangleright (mm').\end{aligned}$$

We note also that $a \triangleright 1 = \varepsilon_t(a)$ and that

$$a \triangleright (a' \triangleright m) = (aa') \triangleright m$$

by the homomorphism and anti-homomorphism properties of Δ and S. Finally, $1 \triangleright m = m$ since both $1_{(1)}$ and $S(1_{(2)})$ belong to V, while $1_{(1)} S(1_{(2)}) = 1_A$. ☐

Theorem 4.6. *The mapping $\phi : m \# a \mapsto ma \in M_1$ defines an isomorphism between the algebra M_1 and the smash product algebra $M \# A$.*

Proof. That ϕ is a linear isomorphism follows from Lemma (2.4).

That ϕ is a homomorphism follows from the conjugation formula in Proposition (4.5):

$$am = a_{(1)} m \varepsilon_s(a_{(2)}) = (a_{(1)} \triangleright m) a_{(2)},$$

since for all $a' \in A$: $\varepsilon_s(a') = S(a'_{(1)}) a'_{(2)} \in V = C_{M_1}(M)$. ☐

Proposition 4.7. *For the action of A on M, we have $N = M^A$.*

Proof. If $n \in N$, then for every $a \in A$:

$$a \triangleright n = a_{(1)} n S(a_{(2)}) = \varepsilon_t(a)(1 \triangleright n) = 1_{(1)} \varepsilon_t(a) n S(1_{(2)}) = \varepsilon_t(a) \triangleright n,$$

using the definition of a module algebra over a weak Hopf algebra.

We similarly compute for each $x \in M^A, a \in A$:

$$\begin{aligned} xS(a) &= \varepsilon_s(a_{(1)}) x S(a_{(2)}) \\ &= S(a_{(1)})(a_{(2)} \triangleright x) \\ &= S(a_{(1)})(\varepsilon_t(a_{(2)}) \triangleright x) \\ &= S(a_{(1)}) \varepsilon_t(a_{(2)}) 1_{(1)} x S(1_{(2)}) = S(a)(1 \triangleright x) = S(a)x \end{aligned}$$

From the bijectivity of $S : A \to A$ and $e_1 \in A$, it follows that $e_1 x = x e_1$, so that $x e_1 = e_1 x e_1 = E(x) e_1$, whence $x = E(x) \in N$. ☐

5. Appendix: The Composite basic construction and a Depth 2 example

In this appendix we discuss the two unrelated topics in the title.

Extending the Jones tower in (22) indefinitely to the right via iteration of the basic construction for a subfactor $N \subseteq M$ of positive index λ^{-1}, Pimsner and Popa [23] have shown that the basic construction of the composite conditional expectation

$$F_n := E \circ E_M \circ \ldots \circ E_{M_{n-1}} : \ M_n \to N$$

is isomorphic to M_{2n+1} with Jones idempotent $f_n \in M_{2n+1}$ given by

$$f_n = \lambda^{-n(n+1)/2} (e_{n+1} e_n \cdots e_1)(e_{n+2} e_{n+1} \cdots e_2) \cdots (e_{2n+1} e_{2n} \cdots e_{n+1}). \tag{51}$$

We will prove here that the same is true in the more general algebraic situation where M/N is a strongly separable extension of index λ^{-1}. We do not need a Markov trace here. This appendix is not needed in Sections 3 and 4.

Let $F_{M_n} = E_{M_n} \circ \cdots \circ E_{M_{2n}} : M_{2n+1} \to M_n$.

Proposition 5.1. *The element f_n is an idempotent satisfying the characterizing properties of the basic construction:*

$$M_{2n+1} = M_n f_n M_n,$$
$$f_n x f_n = f_n F_n(x) = F_n(x) f_n, \quad \forall x \in M_n,$$
$$F_{M_n}(f_n) = \lambda^{n+1} 1.$$

Proof. The proof in [23] that $f_n^2 = f_n$, $F_{M_n}(f_n) = \lambda^{n+1}1$ and $f_n F_n(x) = F_n(x) f_n$ is valid here as it only makes use of the e_i-algebra $A_{n,\lambda}$, the subalgebra of M_n k-generated by $e_1, \ldots, e_n$, and an obvious involution on it. Note that the theorem is true for $n = 0$ (where $f_0 = e_1$). Assume inductively that the proposition holds for $n-1$ and less. We use the induction hypothesis in the second step below, and the Pimsner-Popa identities for sets $f_{n-1}M_{2n-1} = f_{n-1}M_{n-1}$ in the fifth step:

$$\begin{aligned}
M_{2n+1} &= M_{2n} e_{2n+1} M_{2n} \\
&= M_{2n-1} e_{2n} M_{2n-1} e_{2n+1} M_{2n-1} e_{2n} M_{2n-1} \\
&= M_{2n-1} e_{2n} e_{2n+1} M_{n-1} f_{n-1} M_{n-1} e_{2n} M_{2n-1} \\
&= M_{2n-2} e_{2n-1} e_{2n} e_{2n+1} M_{2n-2} f_{n-1} M_{2n-2} e_{2n} e_{2n-1} M_{2n-2} \\
&= M_{2n-2} e_{2n-1} e_{2n} e_{2n+1} f_{n-1} e_{2n} e_{2n-1} M_{2n-2} \\
&= \cdots = M_n e_{n+1} \cdots e_{2n+1} f_{n-1} e_{2n} \cdots e_{n+1} M_n = M_n f_n M_n,
\end{aligned}$$

where the last step is by [23, Lemma 2.3].

Let τ^2 denote the shift map of $A_{n,\lambda} \to A_{n+2,\lambda}$ induced by $e_i \mapsto e_{i+2}$. It follows from the induction hypothesis that $\tau^2(f_{n-1})$ is the Jones idempotent for the composite expectation

$$\widehat{F_{n-1}} := E_{M_1} \circ \cdots \circ E_{M_n} : M_{n+1} \to M_1.$$

Let $x \in M_n$ and $x' = E_{M_{n-1}}(x)$. For the computation below, we note that $e_{n+1} x e_{n+1} = x' e_{n+1}$ and by [23, Remark 2.4]:

$$f_n = \lambda^{-n} (e_{n+1} e_n \cdots e_1) \tau^2(f_{n-1}) (e_2 e_3 \cdots e_{n+1}).$$

We compute:

$$\begin{aligned}
& f_n x f_n = \\
&= \lambda^{-2n} (e_{n+1} \cdots e_1) \tau^2(f_{n-1}) (e_2 \cdots e_{n+1}) x' (e_{n+1} \cdots e_1) \tau^2(f_{n-1}) (e_2 \cdots e_{n+1}) \\
&= \lambda^{-2n} (e_{n+1} \cdots e_1) \widehat{F_{n-1}} (e_2 \cdots e_n x' e_{n+1} e_n \cdots e_2 e_1) \tau^2(f_{n-1}) (e_2 \cdots e_{n+1}) \\
&= \lambda^{-n} (e_{n+1} \cdots e_1) E_M \circ \cdots \circ E_{M_{n-1}}(x) e_1 \tau^2(f_{n-1}) (e_2 e_3 \cdots e_{n+1}) \\
&= F_n(x) f_n. \quad \square
\end{aligned}$$

Remark 5.2. It was shown in [19] that if $N \subseteq M$ is a II_1 subfactor of finite index and *arbitrary* finite depth (see [5] for a definition) then there exists $k \geq 0$ such that for all $i \geq k$ subfactors $N \subseteq M_i$ have depth 2. It would be interesting to extend this property to the purely algebraic case (the finite depth property in this setting was defined in [11]).

As a final topic in this appendix we provide examples of depth 2 extensions in the next proposition and corollary.

Proposition 5.3. *Suppose M/N is a weakly irreducible, symmetric, strongly separable extension such that its bimodule projection $E : M \to N$ has dual bases in the centralizer U. Suppose moreover that the center C of U coincides with the center Z of N. Then M/N has depth 2.*

Proof. Let $x_i, y_i \in U = C_M(N)$ be dual bases of E. It follows that $M \cong N \otimes_Z U$ via $m \mapsto E(mx_i) \otimes y_i$. By the symmetry condition on E, E restricted to U is a trace with values in $Z = C$. Then $\lambda x_i \otimes y_i$ is the symmetric separability element and

$$u \mapsto \lambda x_i u y_i$$

gives a C-linear projection of U onto C coinciding with $E|_U$, since U is an Azumaya C-algebra [24, Section 3].

Let $z_i = \lambda^{-1} x_i e_1$ and $w_i = e_1 y_i$ in M_1: these are dual bases of $E_M : M_1 \to M$ by the Basic Construction Theorem. But we see that $z_i, w_i \in A$.

Next we compute that there are dual bases $x'_i, y'_i \in V = C_{M_1}(M)$ for E_M. By the construction of the last paragraph, it follows that E_{M_1} has dual bases in B, whence M/N has depth 2. We let $x'_i = x_j x_i e_1 y_j$ and $y'_i = x_k y_i e_1 y_k$, both in V. It suffices to compute for $a, b \in M$:

$$\begin{aligned} E_M(ae_1 b x'_i) y'_i &= E_M(aE(bx_j x_i) e_1 y_j) y'_i \\ &= \lambda a E(x_i b x_j) y_j x_k y_i e_1 y_k \\ &= \lambda a x_i x_k y_i e_1 y_k b \\ &= a e_1 E(x_k) y_k b = a e_1 b. \end{aligned}$$

Similarly we compute $x'_i E(y'_i a e_1 b) = a e_1 b$ by using the equivalent expressions $x'_i = x_j e_1 x_i y_j$ and $y'_i = x_k e_1 y_i y_k$. ☐

For the next corollary-example, we need a few definitions. An algebra A is *central* if its center is trivial, $Z(A) = k1$. A ring extension M/N is *H-separable* (after Hirata) if there are elements $f_i \in (M \otimes M)^N$ and $u_i \in U = C_M(N)$ such that $e_1 = u_i f_i$, where e_1 again denotes $1 \otimes 1$ in $M \otimes_N M$ [10].

Corollary 5.4. *Suppose M/N is a split H-separable extension of central algebras where U is Kanzaki separable. Then M/N is a depth 2 strongly separable extension.*

Proof. By the results of [28, Theorem 2.1], the center of U is trivial and $N \otimes U \cong M$ via $n \otimes u \mapsto nu$ for $n \in N, u \in U$. But by the hypothesis U has non-degenerate trace $t : U \to k$ with dual bases $x_i, y_i \in U$. It follows that $E : M \to N$ defined by $E(nu) = \lambda n t(u)$, where $\lambda^{-1} = t(1)$, has dual bases in U. The conclusion now follows readily from the proposition. ☐

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